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A closed crack tip model for interface cracks in thermopiezoelectric materials

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Abstract

The interface crack problem of a bimaterial thermopiezoelectric solid was treated by applying the extended version of Stroh's formalism and singular integral equation approach. The interface crack considered is subjected to combined thermal, mechanical and electric loads. Under the applied loading, the interface crack is assumed to be partially opened. Formulation of the problem results in a set of singular integral equations which are solved numerically. The study shows that the contact zone is extremely small in comparison with the crack length. Based on the formulation, some physically meaningful quantities of interest such as stress intensity factors and size of contact zone for a particular material group are analyzed. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Piezoelastic; Interface; Crack-tip; Stress intensity; Thermal stress

1. Introduction

The crack problems with interfaces in dissimilar materials are of paramount importance for many micromechanics models and numerical fracture mechanics. Some analytical solutions were worked out in the past decades. Most of them indicated that they had oscillatory singularity (Williams, 1959; England, 1965; Erdogan, 1965). As was pointed out by England (1965), a physically unreasonable aspect of the oscillatory singularities is that they lead to overlapping near the ends of the crack. To correct this unsatisfactory feature, Comninou (1977) introduced a closed crack tip model. This idea was further addressed by several authors (Comninou and Schmueser, 1979; Rice, 1988; Gautesen and Dundurs, 1988; Deng, 1994) for isotropic elastic materials. Extensions to anisotropic elasticity have been made by Wang and Choi (1983), Anderson (1988) and Lee and Gao (1994). Previous studies revealed that the stress singularities exhibited an inverse square-root and the size of contact zone was very small in comparison with the crack length in a

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tensile field. As for piezoelectric materials, much work has been done in dealing with crack and dislocation problems. Barnett and Lothe (1975) generalized Stroh's six-dimensional framework to an eight-dimensional framework that includes the line charge and the electric potential jump across the slip plane in a piezoelectric solid. Parton (1976) has considered the problem of a finite crack at the interface between two piezoelectric materials subjected to a far field uniform tension. Dökmeci (1980, 1988) presented critical reviews of the work in the area of dynamics and fracture of piezo crystals. Pak (1992) investigated the electroelastic fields and the energy release rate for a finite crack by way of the method of distributed dislocations and electric dipoles. Fil'shtinskii and Fil'shtinskii (1994) developed a Green function for a piecewise-uniform piezocomposite with a crack between the phases. Besides, the following works should also be mentioned in this context (Parton and Kudryavtsev, 1988; Mikhailov and Parton, 1990; Sosa, 1992; Qin and Mai, 1997). However, similar developments to interface cracks in dissimilar thermopiezoelectric materials have not yet been made, to our knowledge. In this study, we will report a general representation of Comninou's interface crack-tip fields for linear thermopiezoelectric media without friction in the contact zone. The problem is finally reduced to a set of singular integral equations which can be solved with numerical methods. Using the formulation developed, some physically meaningful quantities of interest such as stress intensity factors and size of contact zone for a particular material group are analyzed.

2. Basic equations and expressions

In this section, the extended Stroh formalism (Barnett and Lothe, 1975) used for plane piezoelectric material is briefly reviewed. For a complete derivation and discussion the readers may consult the reference cited above. Consider a 2-D thermoelectroelastic solid, where all field quantities are functions of x_1 and x_2 only. For convenience, the shorthand notation introduced by Barnett and Lothe (1975) is adopted in this paper. In the stationary case when no free electric charge, body force and heat source exist, the complete set of governing equations for uncoupled thermoelectroelastic problems are (Mindlin, 1974)

$$
\begin{aligned}
&h_{i,i} = 0 \\
&\Pi_{iJ,i} = 0
\end{aligned}
\tag{1}
$$

together with

$$
\begin{aligned}
&h_i = -k_{ij} T_{,j} \\
&\Pi_{iJ} = E_{iJKm} u_{K,m} - \chi_{iJ} T
\end{aligned}
\tag{2}
$$

in which

$$
\Pi_{iJ} = \begin{cases} \sigma_{ij} & i, J = 1, 2, 3 \\ D_i & J = 4; \quad i = 1, 2, 3 \end{cases}
$$

$$
u_J = \begin{cases} u_j & J = 1, 2, 3 \\ \vartheta & J = 4 \end{cases}
$$

$$\chi_{iJ} = \begin{cases} \gamma_{ij} & i, J = 1, 2, 3 \\ g_i & J = 4; \quad i = 1, 2, 3 \end{cases} \tag{3}$$

$$E_{iJKm} = \begin{cases} C_{ijkm} & i, J, K, m = 1, 2, 3 \\ e_{mij} & K = 4; \quad i, J, m = 1, 2, 3 \\ e_{ikm} & J = 4; \quad i, K, m = 1, 2, 3 \\ -\kappa_{im} & J = K = 4; \quad i, m = 1, 2, 3 \end{cases} \tag{4}$$

where T and h_i are temperature change and the components of heat flux vector, u_i, ϑ, σ_{ij} and D_i are the components of mechanical displacement vector, electric potential, the components of stress tensor and the components of electric displacement vector, C_{ijkm}, e_{ijk} and κ_{ij} are the elastic stiffness, the piezoelectric coefficients, and dielectric constants, and k_{ij}, γ_{ij} and g_i are the coefficients of heat conduction, thermal-stress constants and pyroelectric constants, respectively. A general solution to (1) can be expressed in terms of extended Stroh formalism as Barnett and Lothe (1975) and Hwu (1992)

$$\begin{aligned} T &= g'(z_t) + \overline{g'(z_t)} \\ \mathbf{u} &= \mathbf{A}\mathbf{f}(\mathbf{z}) + \mathbf{c}g(z_t) + \overline{\mathbf{A}\mathbf{f}(\mathbf{z})} + \overline{\mathbf{c}g(z_t)} \end{aligned} \tag{5}$$

with

$$\begin{aligned} \mathbf{A} &= [\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3 \quad \mathbf{A}_4] \\ \mathbf{f}(\mathbf{z}) &= \mathrm{diag}\,[f_1(z_1) f_2(z_2) f_3(z_3) f_4(z_4)] \\ z_t &= x_1 + \tau x_2 \\ z_i &= x_1 + p_i x_2 \end{aligned}$$

in which "Re" stands for the real part of a complex number, the prime (′) denotes differentiation with the argument, g and $\mathbf{f}$ are arbitrary functions to be determined, p_i, τ, $\mathbf{A}$ and $\mathbf{c}$ are constants determined by

$$\begin{aligned} & k_{22}\tau^2 + (k_{12} + k_{21})\tau + k_{11} = 0 \\ & [\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)p_i + \mathbf{T}p_i^2]\mathbf{A}_i = 0 \\ & [\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)\tau + \mathbf{T}\tau^2]\mathbf{c} = \boldsymbol{\chi}_1 + \tau\boldsymbol{\chi}_2 \end{aligned} \tag{6}$$

in which superscript "T" denotes the transpose, $\boldsymbol{\chi}_i$ are 4×1 vectors, and $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are 4×4 matrices defined by

$$\begin{aligned} & \boldsymbol{\chi}_i = \{\gamma_{i1} \quad \gamma_{i2} \quad \gamma_{i3} \quad g_i\}^T, \\ & (\mathbf{Q})_{IK} = E_{1IK1}, \quad (\mathbf{R})_{IK} = E_{1IK2}, \quad (\mathbf{T})_{IK} = E_{2IK2} \end{aligned} \tag{7}$$

The heat flux vector **h** and the stress–electric displacement (SED) Π obtained from (2) can be written as

$$h_i = -(k_{i1}+\tau k_{i2})g''(z_t)-(k_{i1}+\bar{\tau}k_{i2})\overline{g''(z_t)},$$
$$\Pi_{1J} = -\phi_{J,2}, \quad \Pi_{2J} = \phi_{J,1} \tag{8}$$

where $\boldsymbol{\phi}$ is the SED function given as

$$\boldsymbol{\phi} = \mathbf{Bf(z)}+\mathbf{d}g(z_t)+\overline{\mathbf{Bf(z)}}+\overline{\mathbf{d}g(z_t)} \tag{9}$$

with

$$\mathbf{B} = \mathbf{R}^T\mathbf{A}+\mathbf{TAP} = -(\mathbf{QA}+\mathbf{RAP})\mathbf{P}^{-1}$$
$$\mathbf{P} = \text{diag}\,[p_1 \quad p_2 \quad p_3 \quad p_4]$$
$$\mathbf{d} = (\mathbf{R}^T+\tau\mathbf{T})\mathbf{c}-\boldsymbol{\chi}_2 = -(\mathbf{Q}+\tau\mathbf{R})\mathbf{c}/\tau+\boldsymbol{\chi}_1/\tau. \tag{10}$$

Further, one particular importance in the following discussion is the complex matrix $\mathbf{AB}^{-1}$. It is shown by Ting (1986) that the matrix can be expressed by

$$\mathbf{AB}^{-1} = \mathbf{S}-i\mathbf{L}^{-1} \tag{11}$$

where **S** and **L** are real matrices. The matrix **L** is symmetric and positive definite and the matrix **S** is anti-symmetric. As a consequence, the bimaterial matrix **M** can be given as (Ting, 1986)

$$\mathbf{M} = \mathbf{A}_1\mathbf{B}_1^{-1}-\overline{\mathbf{A}_2\mathbf{B}_2^{-1}} = -(\mathbf{W}+i\mathbf{D}) \tag{12}$$

where subscripts "1" and "2" stand for quantities associated with upper material and lower material, respectively, and where

$$\mathbf{W} = \mathbf{S}_2-\mathbf{S}_1, \quad \mathbf{D} = \mathbf{L}_1^{-1}+\mathbf{L}_2^{-1}. \tag{13}$$

Since $\mathbf{S}_1$, $\mathbf{S}_2$ are anti-symmetric and $\mathbf{L}_1$, $\mathbf{L}_2$ are symmetric, the matrix **W** is anti-symmetric and the matrix **D** is symmetric. The inverse matrix of **M** denoted by **N** can be expressed as

$$\mathbf{N} = \tilde{\mathbf{W}}+i\tilde{\mathbf{D}} \tag{14}$$

where $\tilde{\mathbf{W}}$ is anti-symmetric and $\tilde{\mathbf{D}}$ is symmetric.

3. Piezoelectric bimaterials with closed crack tips

Consider a crack of length $2L$ lying in the interface of dissimilar anisotropic piezoelectric media. Let the crack faces be partially closed with frictionless contact in the intervals $(-L, -a)$ and (b, L)

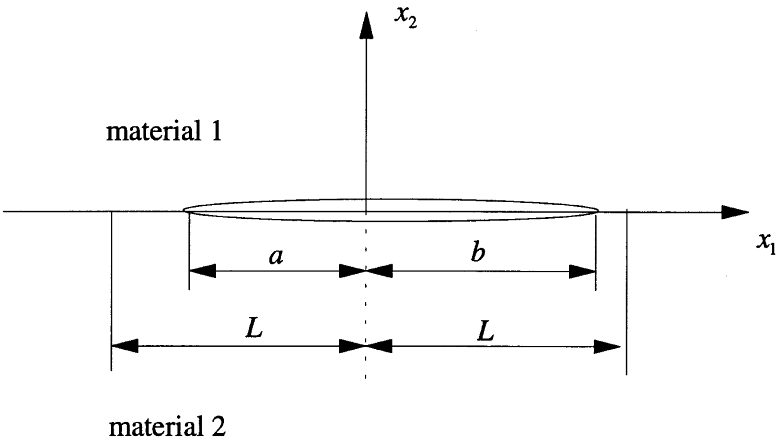

Fig. 1. Coordinates and geometry of a partially closed interface crack.

and opened in the interval $(-a, b)$ (see Fig. 1). We assume that the media is subjected to far field mechanical and electrical loads, say $\mathbf{T}^{\infty}$. The surface of the crack is traction- and charge-free. For the present problem, it suffices to consider the associated problem in which the crack surface is subjected to the conditions

$$\begin{aligned}
&h_2^{(1)}(x_1) = h_2^{(2)}(x_1) = -h^{\infty} \quad (|x_1| < L)\\
&b_K(x_1) = u_K(x_1, 0^+) - u_K(x_1, 0^-) \quad (|x_1| < L; K \neq 2)\\
&b_2(x_1) = u_2(x_1, 0^+) - u_2(x_1, 0^-) \quad (-a < x_1 < b)\\
&\Pi_{2I}^{(1)}(x_1) = \Pi_{2I}^{(2)}(x_1) = -T_I^{\infty} \quad (|x_1| < L; I \neq 2)\\
&\Pi_{22}^{(1)}(x_1) = \Pi_{22}^{(2)}(x_1) = -T_2^{\infty} \quad (-a < x_1 < b)
\end{aligned} \tag{15}$$

and the continuous conditions on the interface excluding the region $[-L, L]$:

$$T_1 = T_2, \quad h_2^{(1)} = h_2^{(2)}, \quad \mathbf{u}^{(1)} = \mathbf{u}^{(2)}, \quad \mathbf{\Pi}_2^{(1)} = \mathbf{\Pi}_2^{(2)} \tag{16}$$

The equality of traction and surface charge continuity comes from the relation $\mathbf{t} = \partial\boldsymbol{\phi}/\partial s$ where $\mathbf{t}$ is the surface traction and charge on a curve boundary and s is the arc-length measured along the curve boundary. When the points along the crack surfaces are considered, integration of $\mathbf{t}^{(1)} = \mathbf{t}^{(2)}$ provides $\boldsymbol{\phi}^{(1)} = \boldsymbol{\phi}^{(2)}$ since the integration constants can be neglected, which correspond to rigid motion. Thus we have

$$T_1 = T_2, \quad h_2^{(1)} = h_2^{(2)}, \quad \mathbf{u}^{(1)} = \mathbf{u}^{(2)}, \quad \boldsymbol{\phi}^{(1)} = \boldsymbol{\phi}^{(2)}. \tag{17}$$

Using eqn $(8)_1$, the continuity condition $(17)_2$ leads to

$$-ik_1 g_1''(x_1^+) - ik_2 \overline{g_2''}(x_1^-) = ik_2 g_2''(x_1^-) - ik_1 \overline{g_1''}(x_1^+) \tag{18}$$

where $x_1^{\pm}$ denote the points on the upper and lower surfaces of the interface, respectively,and $k = k_{22}(\tau - \bar{\tau})/2i$. Noting that the important properties of holomorphic functions used in the method of analytical continuation is that if $g(z)$ is holomorphic in the region $x_2 > 0$ (or $x_2 < 0$), then $\overline{g(\bar{z})}$ is holomorphic in the region $x_2 < 0$ (or $x_2 > 0$). From this property and eqn (18), define

$$\omega(z) = \begin{cases} -ik_1 g_1''(z) - ik_2 \overline{g_2''(z)} & z \in \text{material 1} \\ -ik_2 g_2''(z) - ik_1 \overline{g_1''(z)} & z \in \text{material 2} \end{cases} \tag{19}$$

where the function $\omega(z)$ is analytic in the whole plane. Thus, from Liouville's theorem, we have $\omega(z) = 0$ in our problem. Further if the temperature field tends to zero when $|z| \to \infty$, and the terms corresponding to the rigid body motion are neglected, we have

$$\begin{aligned} \overline{g_1(z)} &= -k_2 g_2(z)/k_1, \quad z \in \text{material 2} \\ \overline{g_2(z)} &= -k_1 g_1(z)/k_2, \quad z \in \text{material 1} \end{aligned} \tag{20}$$

Similarly, from eqns (9) and $(17)_4$, one can obtain

$$\begin{aligned} \overline{\mathbf{f}_1(z)} &= \bar{\mathbf{B}}_1^{-1}\left[\mathbf{B}_2\mathbf{f}_2(z) + \left(\mathbf{d}_2 + \frac{k_2}{k_1}\mathbf{d}_1\right)g_2(z)\right], \quad z \in \text{material 2} \\ \overline{\mathbf{f}_2(z)} &= \bar{\mathbf{B}}_2^{-1}\left[\mathbf{B}_1\mathbf{f}_1(z) + \left(\mathbf{d}_1 + \frac{k_1}{k_2}\mathbf{d}_2\right)g_1(z)\right], \quad z \in \text{material 1} \end{aligned} \tag{21}$$

Using the results of (20) and (21), the continuity conditions $(17)_{1,3}$ can be rewritten as

$$\theta^+ = \theta^-, \quad \boldsymbol{\psi}^+ = \boldsymbol{\psi}^- \quad x_1 \notin [-L, L] \tag{22}$$

where superscripts "+" and "−" indicate the limit values of the corresponding functions as $x_2 \to 0^+$ and $x_2 \to 0^-$, respectively, and

$$\theta(z) = \begin{cases} (1 + k_1/k_2)g_1(z) & z \in \text{material 1} \\ (1 + k_2/k_1)g_2(z) & z \in \text{material 2} \end{cases} \tag{23}$$

$$\boldsymbol{\psi} = \begin{cases} \mathbf{M}\mathbf{B}_1\mathbf{f}_1(z) + \dfrac{1}{k_1 + k_2}(\mathbf{c}^* - \bar{\mathbf{A}}_2\mathbf{B}_2^{-1}\mathbf{d}^*)\theta(z) & z \in \text{material 1} \\ -\bar{\mathbf{M}}\mathbf{B}_2\mathbf{f}_2(z) + \dfrac{1}{k_1 + k_2}(\bar{\mathbf{c}}^* - \bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1}\bar{\mathbf{d}}^*)\theta(z) & z \in \text{material 2} \end{cases} \tag{24}$$

with

$$\mathbf{c}^* = \mathbf{c}_1 k_2 + \bar{\mathbf{c}}_2 k_1, \quad \mathbf{d}^* = \mathbf{d}_1 k_2 + \bar{\mathbf{d}}_2 k_1 \tag{25}$$

The above definitions are quite similar to those of Hwu (1992), except some multipliers which make our derivations easier.

Using the results of eqns (20), (21), (24) and (25), the boundary conditions $(15)_{1,4,5}$ can be expressed as

$$(\theta'')^{+}+(\theta'')^{-}=\frac{-i(k_1+k_2)}{k_1k_2}h^{\infty}$$

$$\mathbf{N}\boldsymbol{\Phi}^{+}-\bar{\mathbf{N}}\boldsymbol{\Phi}^{+}=-\mathbf{T}^{\infty}+\mathbf{F}_{01}(x_1^{+})+\mathbf{F}_{02}(x_1^{-}) \tag{26}$$

for a full open interface crack model, where

$$\boldsymbol{\Phi}=\boldsymbol{\psi}',$$

$$\mathbf{F}_{01}(x_1^{+})=\frac{1}{k_1+k_2}[\mathbf{N}(\mathbf{c}^{*}-\overline{\mathbf{A}_2\mathbf{B}_2^{-1}}\mathbf{d}^{*})-k_2\mathbf{d}_1]\theta(x_1^{+})$$

$$\mathbf{F}_{02}(x_1^{-})=-\frac{1}{k_1+k_2}[\bar{\mathbf{N}}(\bar{\mathbf{c}}^{*}-\overline{\mathbf{A}_1\mathbf{B}_1^{-1}}\bar{\mathbf{d}}^{*})-k_1\mathbf{d}_2]\theta(x_1^{-}) \tag{27}$$

The solution to $(26)_1$ has been discussed elsewhere (see Hwu, 1992, for example). For conciseness we omit those details here.

It should be pointed out that the physical meaning of $\boldsymbol{\Phi}$ is quite obvious and can be seen by noting that the dislocation density vector $\mathbf{b}$ defined by

$$\mathbf{b}=\frac{\partial}{\partial x_1}(\mathbf{u}^{+}-\mathbf{u}^{-}) \tag{28}$$

and checking eqns (24) and $(27)_1$, with which we have

$$\mathbf{b}=(\boldsymbol{\Phi}^{+}-\boldsymbol{\Phi}^{-}) \tag{29}$$

The Plemelj–Sokhotskii formulae for Cauchy integral yields

$$\boldsymbol{\Phi}(z)=\frac{1}{2i\pi}\int_{-L}^{L}\frac{\mathbf{b}}{x_1-z}\mathrm{d}x_1 \tag{30}$$

Introducing the following symbols

$$\mathbf{N}_{*}=\begin{bmatrix}N_{11} & N_{13} & N_{14}\\ N_{31} & N_{33} & N_{34}\\ N_{41} & N_{43} & N_{44}\end{bmatrix},\quad \boldsymbol{\Phi}_{*}=\begin{Bmatrix}\Phi_1\\ \Phi_3\\ \Phi_4\end{Bmatrix},\quad \mathbf{T}_{*}=\begin{Bmatrix}-T_1^{\infty}+(\mathbf{F}_{01})_1+(\mathbf{F}_{02})_1\\ -T_3^{\infty}+(\mathbf{F}_{01})_3+(\mathbf{F}_{02})_3\\ -T_4^{\infty}+(\mathbf{F}_{01})_4+(\mathbf{F}_{02})_4\end{Bmatrix},$$

$$T_{2*}=-T_2^{\infty}+(\mathbf{F}_{01})_2+(\mathbf{F}_{02})_2,\quad \mathbf{W}_0=\{\tilde{W}_{21}\quad \tilde{W}_{23}\quad \tilde{W}_{24}\},\quad \mathbf{D}_0=\{\tilde{D}_{21}\quad \tilde{D}_{23}\quad \tilde{D}_{24}\}$$

$$\mathbf{f}_{*}=-\mathbf{W}_0^{T}(\Phi_2^{+}-\Phi_2^{-})-i\mathbf{D}_0^{T}(\Phi_2^{+}+\Phi_2^{-}),$$

$$f_{2*}=\mathbf{W}_0(\boldsymbol{\Phi}_{*}^{+}-\boldsymbol{\Phi}_{*}^{-})-i\mathbf{D}_0(\boldsymbol{\Phi}_{*}^{+}+\boldsymbol{\Phi}_{*}^{-}) \tag{31}$$

and using eqn (26), the boundary conditions $(15)_{4,5}$ can be written as

$$\mathbf{N}_{*}\boldsymbol{\Phi}_{*}^{+}-\bar{\mathbf{N}}_{*}\boldsymbol{\Phi}_{*}^{-}=\mathbf{T}_{*}+\mathbf{f}_{*}\quad (|x_1|<L) \tag{32}$$

$$N_{22}\Phi_2^{+}-\bar{N}_{22}\Phi_2^{-}=T_{2*}+f_{2*}\quad (-a<x_1<b) \tag{33}$$

Equation (32) is a generalized Hilbert problem if Φ_2 is a known function. The Hilbert problem can be solved by way of the technique of Clements (1971). In doing this, multiplying (32) by Y_i and summing over i results in

$$Y_i N_{*ij}(\Phi_{*j})^+ - Y_i \bar{N}_{*ij}(\Phi_{*j})^- = Y_i(T_{*i}+f_{*i}) \quad (|x_1| < L) \tag{34}$$

The Y_i are chosen such that

$$Y_i N_{*ij} = V_j, \quad Y_i \bar{N}_{*ij} = \lambda V_j \tag{35}$$

Eliminating V_j from (35), one has

$$(\bar{\mathbf{N}}_* - \lambda \mathbf{N}_*)\mathbf{Y} = 0 \tag{36}$$

For a non-trivial solution it is necessary that

$$\|(\bar{\mathbf{N}}_* - \lambda \mathbf{N}_*)\| = 0 \tag{37}$$

Let λ_γ ($\gamma = 1, 2, 3$) denote the roots of eqn (37) and the corresponding values of Y_i and V_i be expressed by $Y_{\gamma i}$ and $V_{\gamma i}$. Equation (34) may then be recast as

$$\{Y_{\gamma i}(\Phi_{*i})^+(x_1)\} - \lambda_\gamma\{Y_{\gamma i}(\Phi_{*i})^-(x_1)\} = Y_{\gamma i}\{T_{*i}(x_1)+f_{*i}(x_1)\} \quad (|x_1| < L) \tag{38}$$

Equation (38) is a standard Hilbert problem and its solution may be given by

$$\Phi_{*i}(z) = \sum_{k=1}^{3} (\mathbf{V}^{-1})_{ik} \left[\frac{X_k(z)}{2\pi i} \int_{-L}^{L} \frac{Y_{kj} T_{*j}(x)}{X_k^+(x)(x-z)} \mathrm{d}x + \frac{X_k(z)}{2\pi i} \int_a^b \frac{Y_{kj} f_{*j}(x)}{X_k^+(x)(x-z)} \mathrm{d}x \right] \tag{39}$$

where

$$X_k(z) = (z-L)^{\delta_k - 1}(z+L)^{-\delta_k}$$
$$\delta_k = \frac{1}{2\pi i} \ln \lambda_k \tag{40}$$

In eqn (39) the branches of $X_k(z)$ are chosen such that $zX_k(z) \to 1$ as $|z| \to \infty$, and the argument of λ_k is selected to lie between 0 and 2π. Substitution of (39) into (33) leads to

$$A(x)b_2(x) + \frac{1}{\pi}\int_{-a}^{b} \frac{B(x,t)}{t-x} b_2(t)\,\mathrm{d}t + \frac{1}{\pi}\int_{-a}^{b} \frac{C(x,\tau)}{(\tau - x)} \left[\frac{1}{\pi}\int_{-a}^{b} \frac{b_2(t)}{t-\tau}\mathrm{d}t\right]\mathrm{d}\tau = F(x) \tag{41}$$

where $A(x)$, $B(x,t)$, $C(x,\tau)$ and $F(x)$ are known functions given in Appendix I. In addition, the solution to (41) should satisfy the separation condition and the condition of unilateral constraint which requires that

$$u_2^{(1)}(x_1, 0) - u_2^{(2)}(x_1, 0) \geqslant 0 \quad (\text{for } -a \leqslant x_1 \leqslant b)$$
$$\sigma_{22}(x_1, 0) \leqslant 0 \quad (\text{for } -L < x_1 \leqslant -a, b \leqslant x_1 < L) \tag{42}$$

For convenience, normalizing the interval $(-a, b)$ by the change of variables;

$$t = \frac{b-a}{2} + \frac{b+a}{2}s, \quad x = \frac{b-a}{2} + \frac{b+a}{2}s_0,$$
$$\tau = \frac{b-a}{2} + \frac{b+a}{2}s_\tau \tag{43}$$

If we retain the same symbols for the new functions, eqn (41) can be rewritten as

$$A(s_0)b_2(s_0) + \frac{1}{\pi}\int_{-1}^{1} \frac{B(s_0, s)}{s-s_0} b_2(s)\,\mathrm{d}s + \frac{1}{\pi}\int_{-1}^{1} \frac{C(s_0, s_\tau)}{(s_\tau - s_0)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{b_2(s)}{s-s_\tau}\mathrm{d}s\right]\mathrm{d}s_\tau = F(s_0) \tag{44}$$

In addition to eqn (44), the single valuedness of elastic displacements and electric potential around a closed contour surrounding the whole crack requires that

$$\int_{-1}^{1} b_2(s)\,\mathrm{d}s = 0 \tag{45}$$

In order to solve the singular integral equations (44) and (45), the numerical method developed by Erdogan and Gupta (1972) is adopted. Let

$$b_2(s) = \frac{\Theta(s)}{\sqrt{1-s^2}} \approx \frac{\sum_{k=1}^{n} B_k T_k(s)}{\sqrt{1-s^2}} \tag{46}$$

where $\Theta(t)$ is a regular function defined in a closed interval $|s| \leqslant 1$, B_k are the real unknown coefficients, and $T_k(t)$ the Chebyshev polynomials of the first kind. Then for $-L < x < a$ and $b < x < L$,

$$\sigma_{22} = A(s_0)b_2(s_0) + \frac{1}{\pi}\int_{-1}^{1} \frac{B(s_0, s)}{s-s_0} b_2(s)\,\mathrm{d}s + \frac{1}{\pi}\int_{-1}^{1} \frac{C(s_0, s_\tau)}{(s_\tau - s_0)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{b_2(s)}{s-s_\tau}\mathrm{d}s\right]\mathrm{d}s_\tau - F(s_0) \tag{47}$$

Since the contact is smooth, σ_{22} should be equal to zero at $s_0 = \pm 1$, i.e. $\sigma_{22} = 0$, which provides two conditions for determining the unknowns, a and b. The discretized form of (44) and (45) may be written as (Erdogen and Gupta, 1972):

$$A(s_{0r})\Theta(s_{0r}) + \sum_{k=1}^{n}\frac{1}{n}\left[\frac{B(s_{0r}, s_k)}{(s_k - s_{0r})} + \frac{1}{m}\sum_{j=1}^{m}\frac{C(s_{0r}, s_{\tau j})(1-s_{\tau j}^2)^{1/2}}{(s_{\tau j} - s_{0r})(s_k - s_{\tau j})}\right]\Theta(s_k) = F(s_{0r})$$
$$\sum_{k=1}^{n}\Theta(s_k) = 0 \tag{48}$$

where

$$s_k = \cos\left[\frac{(2k-1)\pi}{2n}\right], \quad (k = 1, 2, \ldots, n)$$

$$s_{0r} = \cos(r\pi/n), \quad (r = 1, 2, \ldots, n-1)$$

$$s_{\tau j} = \cos\left[\frac{(2j-1)\pi}{2m}\right] \quad (j = 1, 2, \ldots, m, m \neq n)$$

Equation (48) provides a system of n linear algebraic equations to determine $\Theta(s_k)$, and then B_k. Once the function $\Theta(s)$ has been found, the corresponding SED, Π_2 can be given from (8) in the form

$$\Pi_2 = \tilde{\mathbf{W}}\mathbf{b} + \frac{\tilde{\mathbf{D}}}{\pi}\int_{-1}^{1}\frac{\mathbf{b}(s)\,\mathrm{d}s}{s-x} + \mathbf{T}^{\infty} - \mathbf{F}_{01} - \mathbf{F}_{02} \tag{49}$$

where $\Pi_2 = \{\sigma_{21} \quad \sigma_{22} \quad \sigma_{23} \quad D_2\}^T$. With the usual definition, the stress intensity factors are given by

$$\mathbf{K} = \lim_{x_1 \to \pm L}\sqrt{2\pi(x_1 \mp L)}\Pi_2 = \lim_{x_1 \to \pm L}\sqrt{2\pi(x_1 \mp L)}\left[\tilde{\mathbf{W}}\mathbf{b} + \frac{\tilde{\mathbf{D}}}{\pi}\int_{-1}^{1}\frac{\mathbf{b}(s)\,\mathrm{d}s}{s-x} + \mathbf{T}^{\infty} - \mathbf{F}_{01} - \mathbf{F}_{02}\right] \tag{50}$$

where $\mathbf{K} = \{K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}\}^T$. The energy release rate can be evaluated by considering the relative displacements $\Delta\mathbf{U}$, which are obtained from the definition of $\boldsymbol{\psi}$ (24) as

$$\Delta u_i = \sum_{k=1}^{3}(\mathbf{V}^{-1})_{ik}\left\{Y_{kj}\int T_{*j}(s)\,\mathrm{d}s + N_k^{*}\sqrt{L^2-x^2}\sum_{m=1}^{n}\frac{B_m}{m}U_{m-1}(x/L) - N_k^{**}\left[\sum_{m=1}^{n}\frac{B_m}{m+1}T_{m+1}(x/L) - \sum_{m=1,3,5}\frac{B_m}{m+1}\right]\right\} \tag{51}$$

where $U_j(x)$ is the Chebyshev polynomials of the second kind. By applying the virtual work of crack closure method (Irwin, 1957), the total energy release rate G can then be calculated as

$$G = \lim_{\Delta L \to 0}\frac{1}{2\Delta L}\int_0^{\Delta L}\Delta\mathbf{u}^T(x-\Delta L)\Pi_2(x)\,\mathrm{d}x \tag{52}$$

4. Numerical examples

As an illustration of the proposed formulation a simple interface crack problem has been considered (see Fig. 1). The upper and lower materials are assumed to be $BaTiO_3$ (Dunn, 1993) and Cadmium Selenide (1994), respectively. The material constants for the two materials are given in Appendix II.

In our analysis, to be consistent with the notations of Dunn (1993), the plane strain deformation is assumed and the crack line is assumed to be in the x_1–x_3 plane, i.e., $D_2 = u_2 = 0$. Therefore the

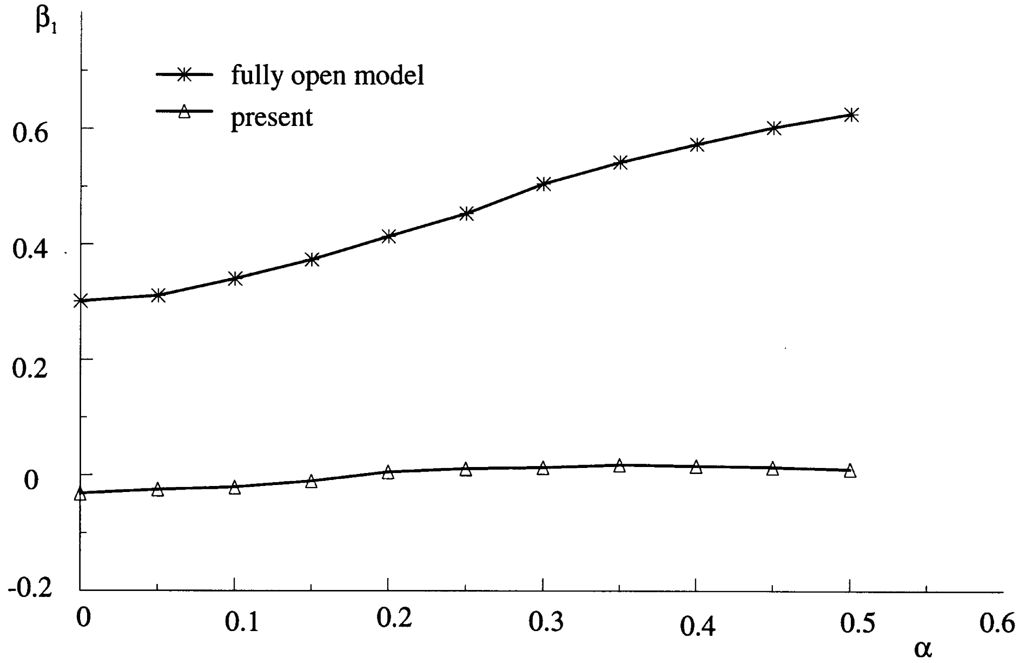

Fig. 2. Stress intensity factors vs parameter α.

stress intensity factor vector $\mathbf{K}^*$ now has only three components $(K_{\mathrm{III}}, K_{\mathrm{I}}, K_{\mathrm{D}})$. In the course of calculation, values of a and b are assumed first, and then computing the n-values of $\Theta(s_k)$ from eqn (48). Once $\Theta(s_k)$ are determined σ_{33} at $s_0 = \pm 1$ can be calculated, and the secant method was applied to predict new values of a and b. This iterative procedure was repeated until determined σ_{33} at $s_0 = \pm 1$ vanishes. Figures 2–4 show the numerical results for the coefficients of stress intensity factors β_{I} $(i = 1, 2, D)$ vs α, where α and β_i are defined by

$$\alpha = h_0 L \gamma_{33} / k_1 \sigma_{13}^{\infty} \tag{53}$$

$$K_{\mathrm{III}}(L) = \sigma_{13}^{\infty} \sqrt{\pi L} \beta_2(\alpha)$$

$$K_{\mathrm{I}}(L) = \sigma_{13}^{\infty} \sqrt{\pi L} \beta_1(\alpha)$$

$$K_{\mathrm{D}}(L) = \sigma_{13}^{\infty} \sqrt{\pi L} \beta_{\mathrm{D}}(\alpha) \tag{54}$$

Figure 2 shows that the heat flux h_0 has very little effect on K_{I} in the closed crack-tip model. However, Figs 3 and 4 show that the intensity factors K_{III} and K_{D} will increase along with the increase of parameter α in some extent. All of these results are compared with those from the fully open model, which will be given in Appendix III. It can be seen from Figs 3 and 4 that there is some discrepancy between the two models, but the discrepancy will gradually abate as the parameter α increases. It should be pointed out that the characteristics of stresses at the crack tip are very different between the fully open model and the closed crack-tip model. Particularly, in the fully open model, the stress σ_{33} becomes unbounded as the point tends to the crack tip, whereas in the

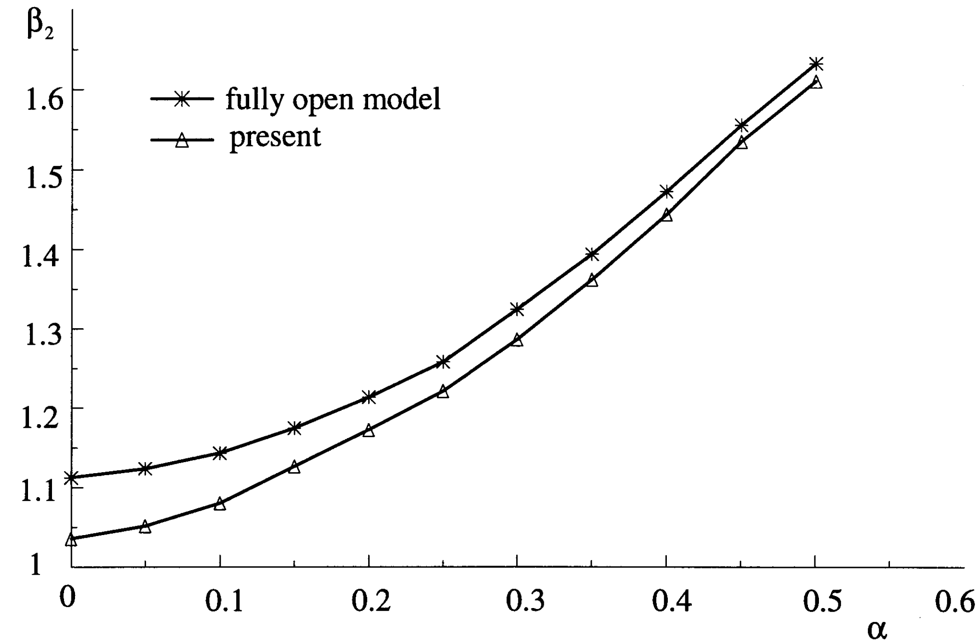

Fig. 3. Stress intensity factors vs parameter α.

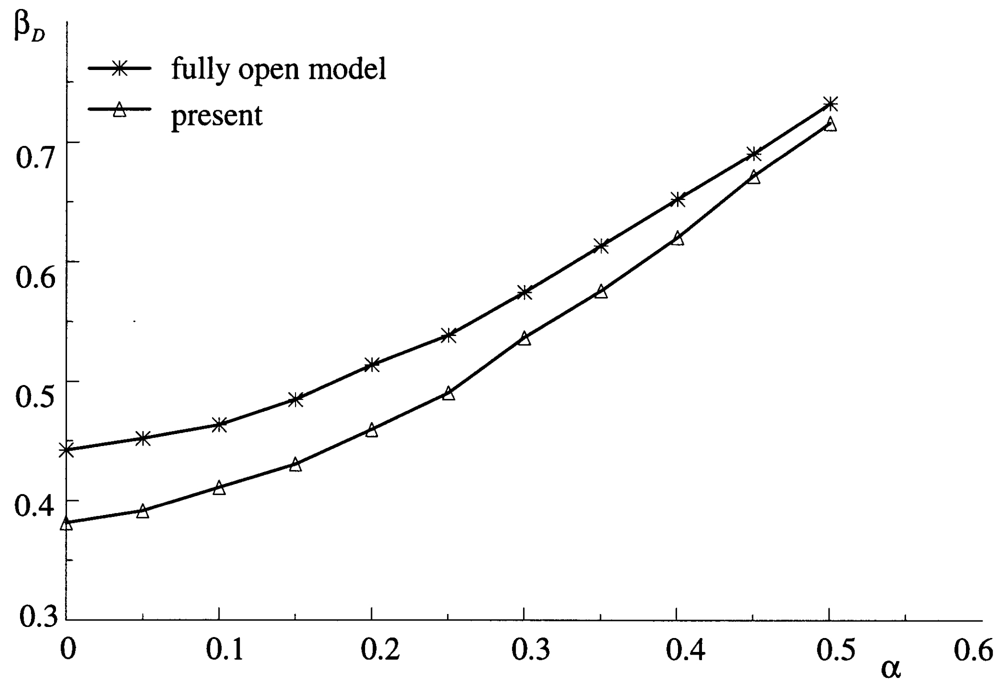

Fig. 4. Stress intensity factors vs parameter α.

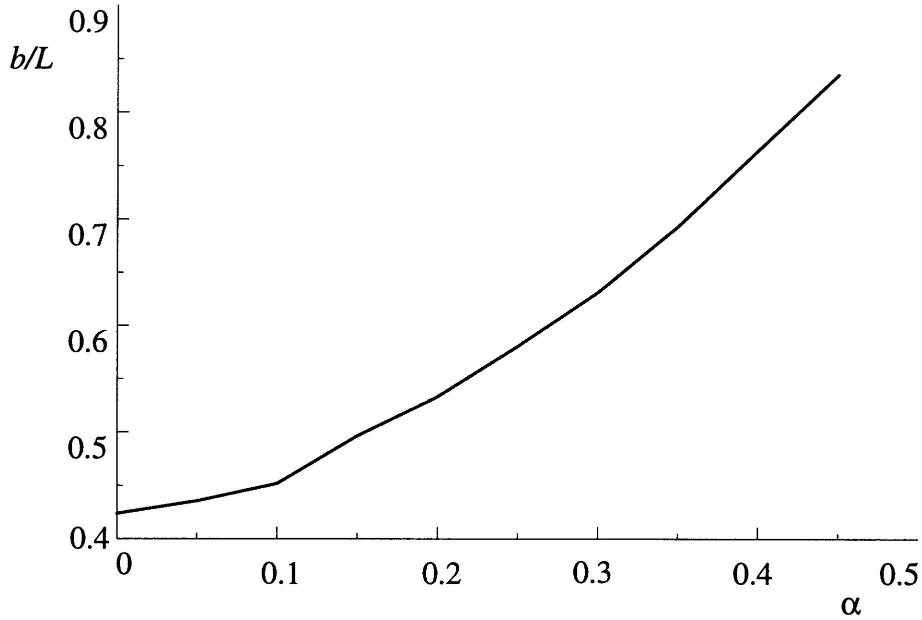

Fig. 5. The right-hand contact zone vs α.

close crack-tip model, the normal stress σ_{33} is finite. Due to this different singular nature, the corresponding results for stress intensity factor K_{I} should be different, but the results for K_{III} and K_{D} should be similar. That is the reason for the large differences between the two models shown in Fig. 2. Further, the numerical results reveal that the left contact zone $(1-a/L)$, is of order 10^{-6} when $\alpha = 0$. We also found that a/L was insensitive to changes of parameter α. For simplicity, the left contact zone was set to be 0.6×10^{-5}. Figure 5 shows the variation of the right-hand contact zone $(1-b/L)$ with parameter α. It can be seen that the length of the contact zone decreases with increasing α. When the shear stress σ_{13}^{∞} is dominant, the contact length of the crack surface becomes significant. Therefore, we can conclude that when the remote heat flux in the x_2 direction is dominant, the contact zone is relatively small and then both models can predict meaningful results. Otherwise, when the shear stress σ_{13}^{∞} is dominant, the contact length becomes significant. In this case the closed crack-tip model should be used to obtain correct results.

5. Conclusions

The extended Stroh formalism and the method of singular integral equation are used to establish a closed crack tip model of an interface crack between two thermopiezoelectric materials. The problem is first reduced to a Hilbert problem and then transformed into a singular integral equation. The equation can be solved numerically. The numerical results show that heat flux h_0 has a small effect on K_{I} for the closed crack tip model, but some considerable effects on the other two concentration factors, i.e. K_{III} and K_{D}. It also can be seen from Figs 3 and 4 that there is some discrepancy between the two models, but the discrepancy decreases with α. Moreover, the numerical results reveal that the left contact zone $(1-a/L)$, is of order 10^{-6} when $\alpha = 0$ and is insensitive to changes of parameter α. From the results reported in Fig. 5, we can conclude that when the remote heat flux in the x_2 direction is dominant, the contact zone is relatively small and then both models can predict meaningful results; otherwise, when the shear stress σ_{13}^{∞} is dominant, the contact length becomes significant. In this case the closed crack-tip model should be used.

Appendix I: The expressions for $A(x)$, $B(x,t)$, $C(x,\tau)$ and $F(x)$

$$A(x) = \sum_{i=1,3,4} \sum_{k=1}^{3} (\mathbf{V}^{-1})_{ik} N_k^* \tilde{W}_{2i} \tag{A1}$$

$$B(x,t) = \tilde{D}_{22} + \sum_{i=1,3,4} \sum_{k=1}^{3} (\mathbf{V}^{-1})_{ik} N_k^{**} \tilde{W}_{2i} - \sum_{i=1,3,4} \sum_{k=1}^{3} \frac{X_k(x)}{X_k^+(t)} (\mathbf{V}^{-1})_{ik} N_k^* \tilde{D}_{2i} \tag{A2}$$

$$C(x,\tau) = -\sum_{i=1,3,4} \sum_{k=1}^{3} (\mathbf{V}^{-1})_{ik} X_k(x) N_k^{**} \tilde{D}_{2i} / X_k^+(\tau) \tag{A3}$$

$$F(x) = T_{2*}(x) + \sum_{i=1,3,4} \sum_{k=1}^{3} (\mathbf{V}^{-1})_{ik} Y_{kj} \left\{ T_{*j} \tilde{W}_{2i} - \frac{1}{\pi} X_k(x) \tilde{D}_{2i} \int_{-L}^{L} \frac{T_{*j}(t)\,\mathrm{d}t}{X_k^+(t)(t-x)} \right\} \tag{A4}$$

where

$$N_k^* = Y_{k1} \tilde{W}_{21} + Y_{k2} \tilde{W}_{23} + Y_{k3} \tilde{W}_{24} \tag{A5}$$

$$N_k^{**} = Y_{k1} \tilde{D}_{21} + Y_{k2} \tilde{D}_{23} + Y_{k3} \tilde{D}_{24} \tag{A6}$$

Appendix II: The material constants used in Section 4

(1) Material properties for $BaTiO_3$ (Dunn, 1993)

$C_{1111} = 150\,\text{GPa},\ C_{1122} = 66\,\text{GPa},\ C_{1133} = 66\,\text{GPa},\ C_{3333} = 146\,\text{GPa},\ C_{2323} = 44\,\text{GPa},$

$\alpha_{11} = 8.53 \times 10^{-6}/\text{K},\ \alpha_{33} = 1.99 \times 10^{-6}/\text{K},\ \lambda_3 = 0.133 \times 10^5\ \text{N/CK},$

$e_{311} = -4.35\,\text{C/m}^2,\ e_{333} = 17.5\,\text{C/m}^2,\ e_{113} = 11.4\,\text{C/m}^2,\ \kappa_{11} = 1115\kappa_0,$

$\kappa_{33} = 1260\,\kappa_0,\ \kappa_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{Nm}^2$

(2) Material properties for Cadmium Selenide (Ashida et al., 1994)

$C_{1111} = 74.1\,\text{GPa},\ C_{1122} = 45.2\,\text{GPa},\ C_{1133} = 39.3\,\text{GPa},$

$C_{3333} = 83.6\,\text{GPa},\ C_{2323} = 13.2\,\text{GPa},$

$\gamma_{11} = 0.621 \times 10^6\ \text{NK}^{-1}\,\text{m}^{-2},\ \gamma_{33} = 0.551 \times 10^6\ \text{NK}^{-1}\,\text{m}^{-2},\ g_3 = -0.294\,\text{CK}^{-1}\,\text{m}^{-2},$

$e_{311} = -0.160\,\text{Cm}^{-2},\ e_{333} = 0.347\,\text{Cm}^{-2},\ e_{113} = 0.138\,\text{Cm}^{-2},$

$$\kappa_{11} = 82.6\times10^{-12}\ \mathrm{C^2/Nm^2},\ \ \kappa_{33} = 90.3\times10^{-12}\ \mathrm{C^2/Nm^2}$$

where α_{ii} and λ_i are thermal expansion constants and pyroelectric constants.

Since the values of the coefficient of heat conduction both for $BaTiO_3$ and Cadmium Selenide could not be found in the literature, the value $k_{33}^{(1)}/k_{11}^{(1)} = 1.5$, $k_{33}^{(2)}/k_{11}^{(2)} = 2$, and $k_{13}^{(1)} = k_{13}^{(2)} = 0$ are assumed.

Appendix III: The fully open crack model

Consider again the crack system shown in Fig. 1 except that we assume the crack is now fully open. In this case eqns (16)–(27) still hold. Obviously eqn $(26)_1$ is a Hilbert problem for a single function, and its solution has been shown in Muskhelishvili (1954) as

$$\theta''(z) = -\frac{k_1+k_2}{2\pi k_1 k_2}\chi_0(z)\int_{-L}^{L}\frac{h^\infty\,\mathrm{d}s}{\chi_0^+(s)(s-z)}+\chi_0(z)p_1(z) \tag{A7}$$

where $p_1(z)$ is a linear function of z, and $\chi_0(z)$ is the basic Plemelj function, i.e., $\chi_0(z) = (z^2-L^2)^{-1/2}$. Having obtained the function θ, the terms in the right-hand side of equation $(26)_2$ are all known. Therefore, it is a Hilbert problem in vector form, and its solution procedure has been discussed in detail elsewhere (see Clements, 1971; Qin and Yu, 1997; for example). Following the technique of Clements (1971), the solution to eqn $(26)_2$ can be expressed as

$$\mathbf{\Phi}(z) = \frac{\mathbf{X}_0(z)}{2\pi i}\int_{-L}^{L}\frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}[-\mathbf{T}^\infty+\mathbf{F}_{01}(x^+)+\mathbf{F}_{02}(x^-)]\,\mathrm{d}s+\mathbf{X}_0(z)\mathbf{p}_1(z) \tag{A8}$$

where $\mathbf{p}_1(z)$ is a vector of linear function, and $\mathbf{X}_0(z)$ a matrix of the basic Plemelj function defined as

$$\mathbf{X}_0(z) = \Lambda\Gamma(z),\quad \Lambda = \mathrm{diag}\,[\lambda_1\quad\lambda_2\quad\lambda_3\quad\lambda_4],\quad \Gamma(z) = \langle(z+L)^{-(1+\delta_k)}(z-L)^{\delta_k}\rangle \tag{A9}$$

where the angular $\langle\ \rangle$ stands for the diagonal matrix, i.e., $\langle A_k\rangle = \mathrm{diag}\,[A_1\quad A_2\quad A_3\quad A_4]$, δ_k and λ_k of (A9) are the eigenvalues and eigenvectors of

$$(\bar{\mathbf{M}}+\mathrm{e}^{2\pi\delta}\mathbf{M})\lambda = 0 \tag{A10}$$

The explicit solution of (A10) has been obtained by Suo et al. (1992) as

$$\delta_{1,2} = 1/2\pm i\varepsilon,\quad \delta_{3,4} = -1/2\pm\kappa \tag{A11}$$

where

$$\varepsilon = \frac{1}{\pi}\tanh^{-1}[(b^2-c)^{1/2}-b]^{1/2},\quad \kappa = \frac{1}{\pi}\tan^{-1}[(b^2-c)^{1/2}+b]^{1/2} \tag{A12}$$

with

$$b = \tfrac{1}{4}\mathrm{tr}\,[(\mathbf{D}^{-1}\mathbf{W})^2],\quad c = |\mathbf{D}^{-1}\mathbf{W}| \tag{A13}$$

here tr stands for the trace of the matrix, and $|\ |$ the determinant of the matrix.

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